CS280 Lecture Notes
L13: Monotone allocations and Myerson's lemma

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1 Recap - Mechanism Design

Mechanism design involves choosing an objective and designing an interaction (a game) such that the outcome of the game achieves the chosen objective. Designing the rules of an auction is one such example. In general, we have three desired *guarantees*:

1. **Dominant Strategy Incentive Compatibilty**, i.e., truthful bidding must be a dominant strategy (**Note:** a strategy is *dominant* if it is atleast as good as any other strategy (for all strategy profiles of the other players))

2. **Social surplus maximisation**, if bidders are truthful, the auction must maximise *surplus*
$$\sum_{i=1}^{n} x_i v_i$$
where x_i is the amount allocated to i

3. **Computational efficiency**, the auction can be implemented in *polynomial time*

Economists usually care about achieving the first two guarantees, whereas we care about the third as well. In situations where achieving all three guarantees is infeasible, we relax one or more and proceed with our investigation. (ex. achieving 90% of maximum surplus might be sufficient).

2 Sponsored Search Auctions

Situation: Suppose you type a query on an e-commerce website like Amazon. The website lists out the results relevant to your query. The *order* of these results is important, as the user is more likely to pay more attention to the ones on top. Most people do not even look beyond the first 2 or 3 pages of results. These are the places the advertisers want to be in, the higher up the better. Behind the scenes, an auction is run to decide which advertisers' links are shown, the order of the links, and how advertisers are charged.

We formalize these ideas below [3].

1. Items for sale here are the k *"slots"*
2. *Bidders* are the advertisers
3. Each slot j has a click-through-rate (CTR) a_j. Click-through-rate of a slot is the probability to get a click on that slot
4. Each bidder i has a private valuation v_i and gets value $a_j \cdot v_i$. We take $a_1 \geq a_2 ... \geq a_k$

Auctioneer chooses k winners and declares slots to them based on their payments. We have demonstrated the need for payments to keep players truthful in the previous lecture.

Definition 1 (Single parameter environments). *A single parameter environment is defined by*

1. *n bidders with private v_i*
2. *Feasible set χ, each element of which is an n-dimensional vector $(x_1, x_2, ..., x_n)$ in which x_i is the amount given to i*

In essence, every bidder has only one valuation. Some examples to illustrate:

- Single-item auctions: χ would be 1-hot vectors with at-most one 1, i.e., $\sum x_i \leq 1$
- Auction of k identical items where each bidder can get atmost one, χ would once again be 0-1 vectors with $\sum x_i \leq k$
- In *sponsored search* discussed above, χ is the set of n-vectores with x_i being a_j if slot j is assigned to bidder i

Definition 2 (Allocation and payments). *A sealed-bid auction is defined by:*

- *Bids $(b_1, ..., b_n)$ reported by bidder*
- *Feasible allocation $x(b) \in \chi$ chosen by the auctioneer*
- *Payments $p(b) \in \mathbb{R}^n$ chosen by the auctioneer*
- *Utility of bidder $i : u_i = v_i \cdot x_i(b) - p_i(b)$*

3 Monotone Allocations and Myerson's Lemma

Definition 3 (Monotone allocations). *An allocation rule x for a single-parameter environment is monotone if for every bidder i and bids b_{-i} by the rest of the bidders, the allocation $x_i(z, b_{-i})$ is non-decreasing in z*

Theorem 1 (Myerson's Lemma). *Let (x, p) be a mechanism. We assume that $p_i(b) = 0$ whenever $b_i = 0$, for all bidders i.*

1. *It holds that if (x, p) is DSIC mechanism then x is **monotone***

2. *If x is a monotone allocation, then there is a unique payment rule such that (x, p) is DSIC*

Proof. **Part 1.** Suppose (x, p) is DSIC and let $0 \leq y \leq z$. We split into two cases.

Case 1 z is the valuation of player i
If i bids z we have

$$u_i = z \cdot x_i(z, b_{-i}) - p_i(z, b_{-i})$$

If i bids y instead, we have

$$u_i^{'} = z \cdot x_i(y, b_{-i}) - p_i(y, b_{-i})$$

From DSIC, we require the truthful bid to be better, i.e., $u_i \geq u_i^{'}$

$$z \cdot x_i(z, b_{-i}) - p_i(z, b_{-i}) \geq z \cdot x_i(y, b_{-i}) - p_i(y, b_{-i}) \tag{1}$$

This case is underbidding, we take up overbidding next!

Case 2 y is the valuation of player i
If i bids y we have

$$u_i = y \cdot x_i(y) - p_i(y)$$

If i bids z instead, we have

$$u_i^{'} = y \cdot x_i(z) - p_i(z)$$

b_{-i} term has been excluded for convenience only. Again from DSIC, we require $u_i \geq u_i^{'}$

$$y \cdot x_i(y) - p_i(y) \geq y \cdot x_i(z) - p_i(z) \tag{2}$$

Adding equations (1) and (2) we get:

$$\begin{aligned}
z \cdot x_i(z) - p_i(z) + y \cdot x_i(y) - p_i(y) &\geq z \cdot x_i(y) - p_i(y) + y \cdot x_i(z) - p_i(z) \\
\implies z \cdot x_i(z) + y \cdot x_i(y) &\geq z \cdot x_i(y) + y \cdot x_i(z) \\
\implies (z - y) \cdot x_i(z) &\geq (z - y) \cdot x_i(y) \\
\implies x_i(z) &\geq x_i(y)
\end{aligned}$$

Hence, x is **monotone**.

□

4 Myerson's Lemma: Payments. (Part 2)

To establish payment differences with respect to bids y and z, we have:

$$z \cdot (x_i(y) - x_i(z)) \leq p(y) - p(z) \leq y \cdot (x_i(y) - x_i(z)). \tag{3}$$

This inequality bounds the rate of change in payments as bids vary Fig.1 (a).

- location is not differentiate
- location is monotone

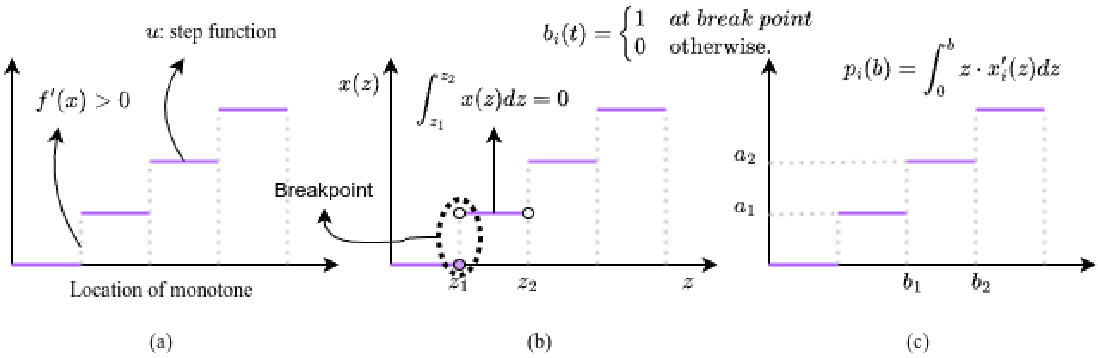

Figure 1: Visualize of Payment

When y and z are close, we take the limit $y \to z$, which yields:

$$z \cdot h \leq p(y) - p(z) \leq y \cdot h$$

There are jump point in Fig.1, which change the $x(z)$ value, so

$$p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j \cdot \in x_i(., b_{-i}) at z_j$$

Only this breakpoint generate the value when we apply the integral Fig.1 (b). Therefore, if we devide top inequality and let $y \to z$ we get below from Eq.3.

$$p'(z) = z \cdot x_i'(z). \tag{4}$$

The payment function can thus be derived by integrating over bids, if x is **differentiable** [2]:

$$p(z) = \int_0^z z \cdot x_i'(z)\, dz.$$

$$p_i(b_i, b_{-i}) = \int_0^{b_i} z \cdot \frac{dx_i(z, b_{-i})}{dz} dz$$

in here $f(b_1) = a_1$ and $f(b_2) = a_2$

$$b_i = \begin{cases} z < b_1 & 0 \\ b_1 < z < b_2 & a_1 \cdot b_1 \\ z > b_2 & a_1 \cdot b_1 + b_2(a_2 - a_1). \end{cases}$$

So, the Myerson‘s Lemma: payments $p_i(b)$ is defined as:

$$p_i(b) = \sum_{j=1}^{k} b_{j+1}(a_j - a_{j+1})$$

5 Prove payment mechanism

In this section, we explore the payment mechanism under Myerson’s Lemma and prove its alignment with monotonicity and optimal payments. The payment function must be structured to satisfy the conditions of monotonicity, feasibility, and

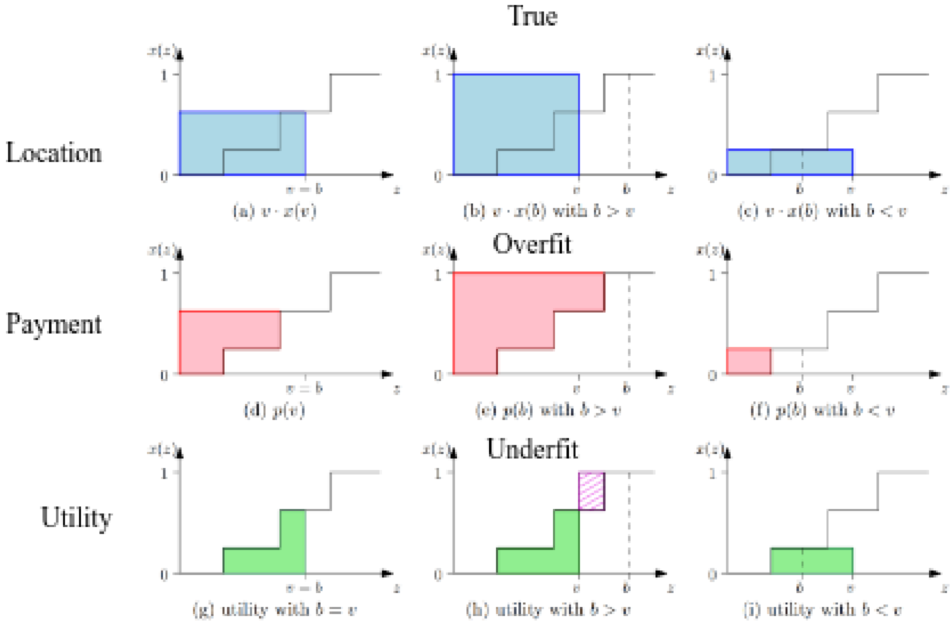

Figure 2: Illustration of the Payment Mechanism in Myerson's Lemma. This figure visualizes the locations, payments, and utilities associated with different bidding scenarios.

incentive compatibility, ensuring that agents maximize utility by truthfully reporting their bids.

In accordance with Myerson's Lemma, payments $p(z)$ must satisfy the monotonicity condition. This ensures that as the bid z increases, the payment does not decrease. Monotonicity is critical to align payments with incentive constraints, guaranteeing that agents benefit most by reporting their bids truthfully.

The proof for this payment mechanism involves three parts:

1. **True, Overfitting, and Underfitting Scenarios:** To visualize the impact of the payment mechanism, we can represent true values, overfitting, and underfitting cases in Fig.2. Each row of this figure would correspond to:
   - Row 1: Locations (indicating the position of bids or breakpoints),
   - Row 2: Payment values (demonstrating adherence to monotonicity),
   - Row 3: Utility outcomes (highlighting the utility maximization achieved by truthful bidding).

2. **Payment Formula:** Using Myerson's Lemma, the payment at each bid z can be calculated based on the cumulative sum of product terms at each breakpoint. This is expressed as:

$$p_i(b_i, b_{-i}) = \sum_{j=1}^{l} z_j \cdot \text{jump in } x_i(\cdot, b_{-i}) \text{ at } z_j,$$

   where z_j are breakpoints, and each term represents a contribution based on the jump in the allocation function x_i at that breakpoint.

3. **Incentive Compatibility and Feasibility:** By ensuring monotonic payments, the mechanism maintains incentive compatibility and feasibility, meaning agents are encouraged to report their bids truthfully without external incentives.

In **sponsored search auctions**, multiple bidders compete for top positions in search results. The **top-k highest bidders** win these positions, with higher bids securing higher ranks. While assigning positions is straightforward, calculating the **payments** each winner should make is more complex. If $p(b_1)$ and $p(b_2)$ represent payments for the top bids, this payment structure ensures fairness and encourages truthful bidding, optimizing the auction's effectiveness [1, 4].

Division of work

- **Gnana Heemmanshuu Dasari:** Beginning to proof of part 1 of Myerson's lemma (Sections 1,2,3)
- **Sungheon Jeong:** Proof of part 2 of Myerson's lemma and more (Section 4 onwards)

References

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